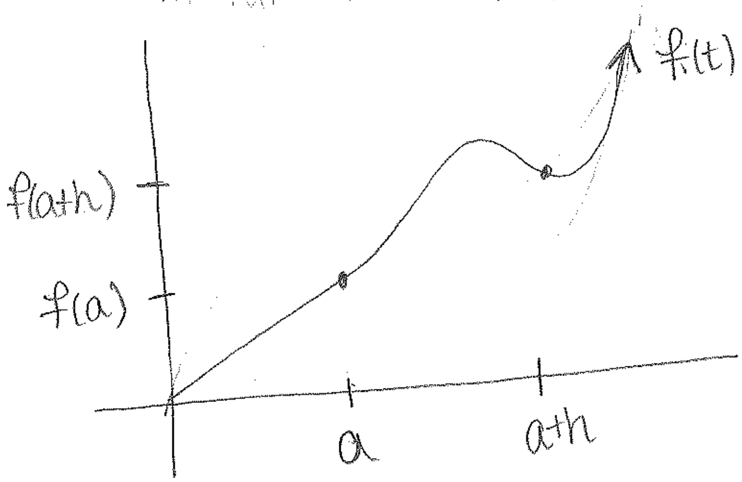
Jan. 13, 2014

A hint for WeBWorK problem 1: Read second half of pg. 81–82

Last Time:

position function: $f(t)$



Avg. velocity from $t=a$ to $t=a+h$:

$$\frac{f(a+h)-f(a)}{(a+h)-a} = \boxed{\frac{f(a+h)-f(a)}{h}}$$

slope of secant line through $(a, f(a))$, $(a+h, f(a+h))$

Instantaneous velocity: the value of $\frac{f(a+h)-f(a)}{h}$ as h "goes to zero". we will see what that means.

Goal: Rates of change in general

$f(x)$ could be distance vs. time x
profit vs supply x
birthrate vs. population x
velocity vs. time x (acceleration)

DEF: Given a function f,
the avg rate of change of f over an interval $[x, x+h]$ is $\frac{f(x+h)-f(x)}{h}$

* the instantaneous rate of change of f at a point x is the "limit" of the avg. rates of change on $[x, x+h]$ as "$h \to 0$" (h goes to zero)

What does that mean? Let's talk limits.

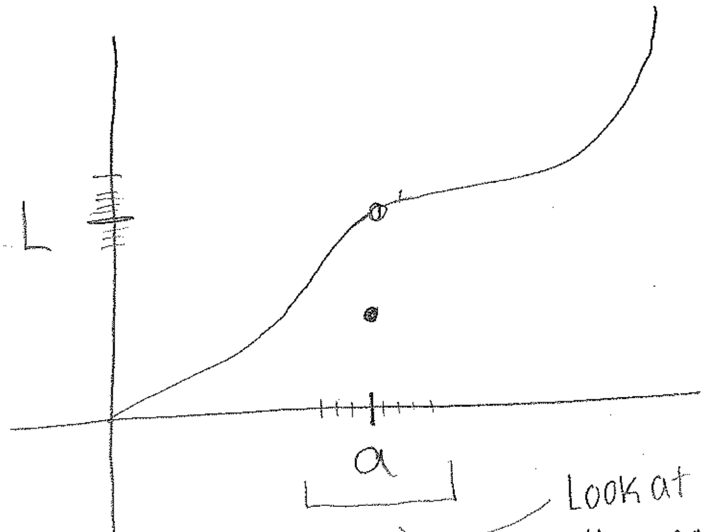
Limit of a Function

"Definition"

We say that a function f "approaches the limit L" as x approaches "a"

written $\lim_{x \to a} f(x) = L$

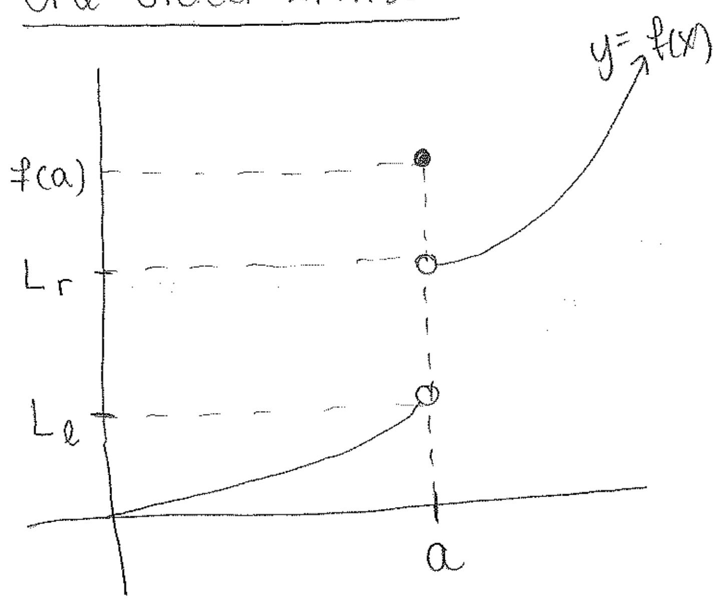
If we can make $f(x)$ as close to L as we want by taking x sufficiently close to a.



Look at x values very "close" to a the corresponding $f(x)$ values will be "close" to L

another thought:
you're traveling along the graph. The limit is the value the graph "appears" to be approaching.

One-Sided Limits



Right-handed Limit: $L_r = \lim_{x \to a^+} f(x)$

If $f(x)$ gets closer to L_r as x gets closer to a from the right

Left-handed Limit: $L_l = \lim_{x \to a^-} f(x)$

If $f(x)$ gets closer to L_l as x gets closer to a from the left.

<u>Theorem</u>: $\lim_{x \to a} f(x)$ exist if and only if
$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

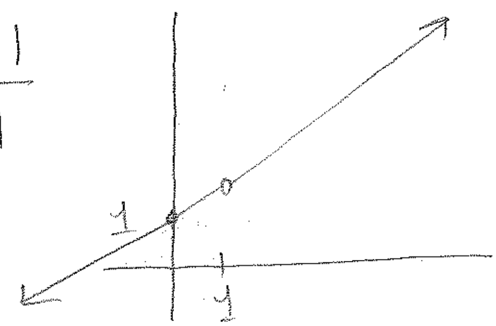
* If you are traveling to a along the graph from different directions, it has to look like you're going to the same place from both directions *

<u>Examples:</u>

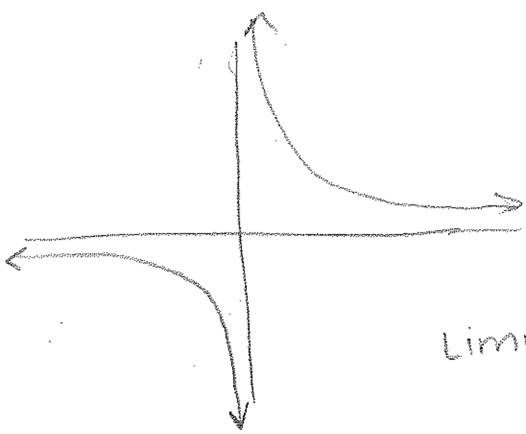
- $\lim_{x \to 2} \frac{x-2}{x+3}$ exists $\quad \lim_{x \to 2^-} \frac{x-2}{x+3} = 0$
$\quad \lim_{x \to 2^+} \frac{x-2}{x+3} = 0$

- $\lim_{x \to 1} \frac{x^2-1}{x-1}$

exists



- $\lim_{x \to 0} \frac{1}{x}$



From the left, getting very negative.

From the right, getting very positive.

Limit does not exist! (DNE)

(Note: limits from left and right will always exist, (unless ∞) just need to worry about $\lim_{x \to a} f(x)$)

Some Nice Things About Limits

If $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$ both exist, then

(1) $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = A + B$

(2) $\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = A - B$

(3) $\lim_{x \to a} (f(x) g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = A \cdot B$

(4) $\lim_{x \to a} \left(f(x) / g(x) \right) = \lim_{x \to a} f(x) \Big/ \lim_{x \to a} g(x) = \dfrac{A}{B}$ $(B \neq 0)$

What does this tell us about taking the limit?

Solving limits:

Step 1: Can you plug "a" into f(x)?
If so, do it! (this is what the properties tell us)

ex: $\lim_{x \to 2} \dfrac{x-2}{x+3} = \dfrac{(2)-2}{(2)+3} = 0$

Step 2: If step 1 doesn't work, can you use algebra to simplify the problem (like cancellation)? If so, do it then plug a into f(x).

ex: $\lim_{x \to 1} \dfrac{x^2-1}{x-1} = \lim_{x \to 1} \dfrac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}} = \lim_{x \to 1} x+1 = (1)+1 = 2$

Step 3: Learn special limits to fix common problems (later)

If in doubt, graph it!

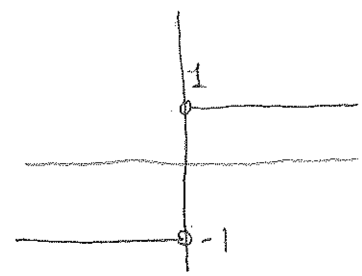
Examples:

(1) $\lim\limits_{x\to 1} \dfrac{x^2-3x+2}{x^2+4x-5} = \lim\limits_{x\to 1} \dfrac{(x-2)\cancel{(x-1)}}{(x+5)\cancel{(x-1)}}$

$= \lim\limits_{x\to 1} \dfrac{x-2}{x+5} = \dfrac{1-2}{1+5} = -\dfrac{1}{6}$

(2) $\lim\limits_{x\to -2} \dfrac{|x|}{x} = \dfrac{2}{-2} = -1$

(3) $\lim\limits_{x\to 0} \dfrac{|x|}{x} = \text{DNE}$



(4) $\lim\limits_{x\to 0} \dfrac{(3+x)^2-3^2}{x} = \lim\limits_{x\to 0} \dfrac{9+6x+x^2-9}{x}$

$= \lim\limits_{x\to 0} \dfrac{6x+x^2}{x} = \lim\limits_{x\to 0} 6+x$

$= 6$

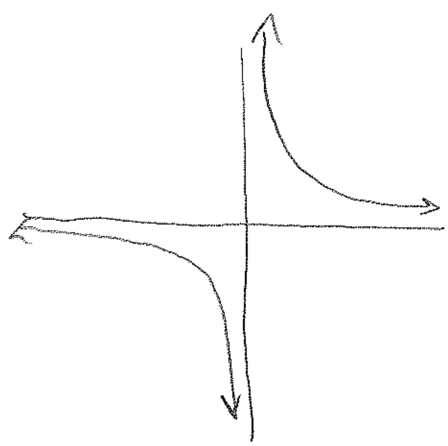
(Write $\lim\limits_{x\to a}$ everytime until you evaluate the limit)

Infinite Limits

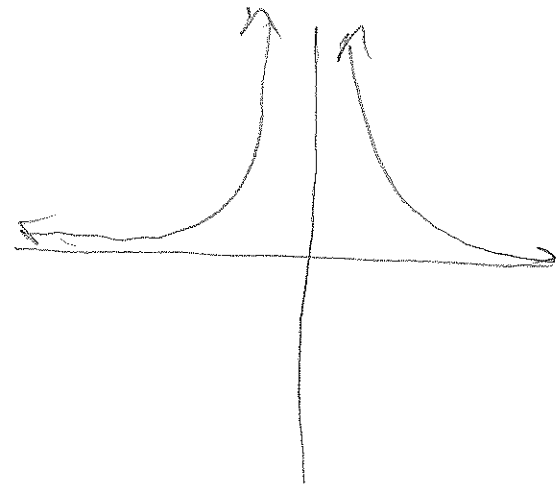
If $f(x)$ gets arbitrarily large as $x \to a$, we like to give more info than DNE.

Ex:

$f(x) = 1/x$



$f(x) = 1/x^2$



$\lim_{x \to 0^-} 1/x = -\infty$

$\lim_{x \to 0^+} 1/x = \infty$

so...

$\lim_{x \to 0} 1/x = \text{DNE}$

$\lim_{x \to 0^-} 1/x^2 = \infty$

$\lim_{x \to 0^+} 1/x^2 = \infty$

so...

$\lim_{x \to 0} 1/x^2 = \infty$

★ ∞ is <u>not</u> a number. You can't do math with it.
You can only use it to tell us what is going on★

Graphically: $\lim_{x \to a^+} f(x) = \pm\infty$; $\lim_{x \to a^-} f(x) = \pm\infty$

means there is a vertical asymptote at $x=a$

Limits at Infinity

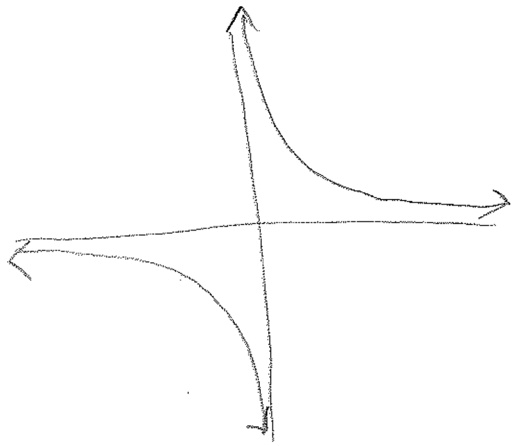
what does

$\lim_{x \to \infty} f(x) = L$ OR $\lim_{x \to -\infty} f(x) = L$ mean?

$\underbrace{\lim_{x \to \infty} f(x) = L}_{\text{as } x \text{ get larger and larger in magnitude, } f(x) \text{ approaches the value } L}$

$\underbrace{\lim_{x \to -\infty} f(x) = L}_{\text{as } x \text{ gets larger and larger in the negative direction, } f(x) \text{ approaches the value } L.}$

Ex: $f(x) = 1/x$



$\lim_{x \to \infty} 1/x = 0$

$\lim_{x \to -\infty} \frac{1}{x} = 0$

Rational Functions

Limits that look like they're going to $\frac{\infty}{\infty}$, could be doing many things:

Ex 1: $\lim_{x \to \infty} \frac{x-1 \to \infty}{x^3+2 \to \infty}$

It matters how "fast" the top and bottom are going to ∞ in relation to each other.

$= \lim_{x \to \infty} \frac{x-1}{x^3+2} \cdot \frac{1/x^3}{1/x^3}$

TRICK: divide top and bottom by largest power in denominator

$= \lim_{x \to \infty} \frac{1/x^2 - 1/x^3}{1 + 2/x^3}$

$\lim_{x \to \infty} \frac{1}{x^p} = 0$ when $p \geq 1$

$= \lim_{x \to \infty} \frac{0 - 0}{1 + 0} = 0$

$\underline{\text{Ex 2:}}$ $\lim_{x\to\infty} \frac{3x^2-2x+1}{4x^2-1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{3-\frac{2}{x}+\frac{1}{x^2}}{4-1/x^2}$

$= \frac{3-0+0}{4-0}$

$= \boxed{3/4}$

$\underline{\text{Ex 3:}}$ $\lim_{x\to\infty} \frac{x^4+x^2+2}{x^3+3} \cdot \frac{1/x^3}{1/x^3}$

$= \lim_{x\to\infty} \frac{x+1/x+2/x^3 \longrightarrow \infty}{1+3/x^3 \longrightarrow 1}$

$= \infty$

Quick Trick:

$P(x) = a_n x^n + \dots + a_1 x + a_0 \qquad Q(x) = b_m x^m + \dots + b_1 x + b_0$

$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\infty} \frac{a_n x^n}{b_m x^m} = \begin{cases} 0 & \text{if } n<m \\ a_n/b_m & \text{if } n=m \\ \pm\infty & \text{if } n>m \end{cases}$

$\underline{\text{Ex:}}$ $\lim_{x\to\infty} \frac{x}{\sqrt{3x^2+2}} \cdot \frac{1/x}{1/x}$ (square root of a square is x

$= \lim_{x\to\infty} \frac{1}{\sqrt{3+\frac{2}{x^2}}} = \frac{1}{\sqrt{3}}$

$\underline{\text{Ex:}}$ $\lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x}$ (mult by conj on top and bottom)

$= \lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x} \cdot \frac{\sqrt{2+x}+\sqrt{2}}{\sqrt{2+x}+\sqrt{2}} = \lim_{x\to 0} \frac{2+x-2}{x(\sqrt{2+x}+\sqrt{2})} = \lim_{x\to 0} \frac{1}{\sqrt{2+x}+\sqrt{2}}$

$= \frac{1}{2\sqrt{2}}$